KERALA UNIVERSITY
Model Question Paper- M. Sc. Examination
Branch : Mathematics
MM232 - FUNCTIONAL ANALYSIS - I

Time: 3 hours | Max. Marks:75

Part A

Answer any 5 questions from among the questions 1 to 8
Each question carries 3 marks

1. Let X_1 be a closed subspace and X_2 be a finite dimensional subspace of a normed space X. Prove that $X_1 + X_2$ is closed in X.
2. Prove that every linear map defined on a finite dimensional space is continuous.
3. Check whether l^p has a denumerable basis. Find a Schauder basis for l^p.
4. Let X be a normed linear space and a be a nonzero element of X. Prove that $\|a\| = \sup\{|f(a)| : f \in X', \|f\| \leq 1\}$
5. Define a closed linear map. Whether every closed linear maps are continuous? Justify the answer.
6. Let X be a Banach space over K and $A \in BL(X)$. Prove that the spectrum of A, $\sigma(A)$ is a compact subset of K.
7. Let X be a finite dimensional space. Prove that $x_n \xrightarrow{w} x$ if and only if $x_n \to x$.
8. Does there exist a compact linear map $F : l^\infty \to l^\infty$ which is onto. $5 \times 3 = 15$

Part B

Anwer all questions from 9 to 13
Each question carries 12 marks

9. A. Let X be a normed linear space. Prove that every closed and bounded subset of X is compact if and only if X is finite dimensional. 4 marks

   B. Let X and Y be normed linear spaces and $F : X \to Y$ be linear. Show that F is continuous if and only if $\|F(x)\| \leq \alpha\|x\| \ \forall x \in X$ and for some $\alpha > 0$. 5 marks

   C. If X is an infinite dimensional normed space, then prove that it contains a hyperspace which is not closed. 3 marks

OR

   A. State and prove Riesz Lemma. 4 marks

   B. Let X and Y be normed spaces and $F : X \to Y$ be linear and range of F be closed. Show that F is continuous if and only if the zero space $Z(F)$ is closed in X. 5 marks

   C. Let $f : X \to c$ defined by $f(x) = \lim_{j\to\infty} x(j),\ x \in c$. Prove that f is continuous and $\|f\| = 1$. 3 marks

10. A. Let X be a normed space over K, Y be a subspace of X and $g \in Y'$. Prove that there exist $f \in X'$ such that $f|_Y = g$ and $\|f\| = \|g\|$. Let $X = K^2$ with norm $\|.\|_\infty$ and $Y = \{(x_1, x_2) : x_2 = 0\}$. Define $g \in Y'$ by $g(x(1), x(2)) = x(1)$. Find a Hahn Banach extension to g. 8 marks

B. Let $a = (1, 1, 1, \ldots)$. Prove that $\{a, e_1, e_2, \ldots\}$ forms a Schauder basis for the subspace c of l^∞. 4 marks

OR

A. Let X be a normed linear space. Prove that for every subspace Y of X and every $g \in Y'$ there exists a unique Hahn Banach extension of g to X if and only if X' is strictly convex. 6 marks

B. Let X and Y be normed spaces and $X \neq \{0\}$. Prove that $BL(X, Y)$ is a Banach space if and only if Y is Banach. Prove that X' is Banach. 6 marks

11. A. State and prove Uniform Boundedness Principle. 6 marks

B. Let X and Y be normed spaces and $F \in BL(X, Y)$. If F is open, prove that F is onto. 3 marks

C. Let X be a Banach space and $P : X \to X$ be a projection. If range of P and zero space of P are closed then prove that P is continuous. 3 marks

OR

A. Let X be a normed space and E be a subset of X. Prove that E is bounded in X if and only if $f(E)$ is bounded in K for every $f \in X'$. 6 marks

B. State and prove Open Mapping theorem. 6 marks

12. A. Let X be a normed space and $A \in BL(X)$ be of finite rank. Prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$. 6 marks

B. Let X and Y be Banach spaces and $F \in BL(X, Y)$ be one-one. If range of F, $R(F)$ is closed in Y, prove that $F^{-1} : R(F) \to X$ is bounded. 3 marks

C. If $A \in BL(X)$ is invertible, prove that $\sigma(A^{-1}) = \{k^{-1} : k \in \sigma(A)\}$. 3 marks

OR

A. Let $A : l^p \to l^p$ defined by $A(x) = (0, x(1), x(2), \ldots)$; $x \in l^p$. Find the spectrum, eigen spectrum and approximate eigenspectrum of A. 6 marks

B. Let X be a nonzero Banach space over C and $A \in BL(X)$. Prove that spectrum of A is nonempty. Obtain the spectral radius formula. 6 marks

13. A. Let X be a reflexive normed space. Prove that X' is reflexive. 4 marks

B. Let $F \in BL(X, Y), G \in BL(Y, Z)$ and one of them be compact. Prove that $GF \in CL(X, Z)$. 4 marks

C. Let X be a Banach space and $P \in BL(X)$ be a projection. Prove that $P \in CL(X)$ if and only if P is of finite rank. 4 marks

OR

A. Let X be a normed linear space and Y be a Banach space. Prove that $CL(X,Y)$ is a closed two sided ideal of $BL(X,Y)$. 6 marks

B. Prove that $x_n \overset{w}{\to} x$ if and only if $x_n \to x$ in l^1. 6 marks

. $5 \times 12 = 60$